REMARKS

Supplemental to the remarks in the Amendment of April 16, 2004, Applicants submit two internet references, the printouts of which are attached, which show that the Intermediate Value Theorem as explained in the previously submitted Amendment, is well known to one skilled in the art:

http://archives.math.utk.edu/visual.calculus/1/continuous.7/

http://oregonstate.edu/instruct/mth251/cq/Stage4/Lesson/IVT.html

In view of the above, reconsideration and allowance of this application are now believed to be in order, and such actions are hereby solicited. If any points remain in issue which the Examiner feels may be best resolved through a personal or telephone interview, the Examiner is kindly requested to contact the undersigned at the telephone number listed below.

The USPTO is directed and authorized to charge all required fees, except for the Issue Fee and the Publication Fee, to Deposit Account No. 19-4880. Please also credit any overpayments to said Deposit Account.

Respectfully submitted,

Randal Kimble Reg # 44,186 for

Seok-Won Stuart Lee*
*Granted limited recognition under 37 C.F.R. § 10.9(b), as shown in a copy of the same filed on April 16, 2004, at the U.S.P.T.O.

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Date: April 26, 2004

Visual Calculus

Properties of Continuous Functions

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Objectives: In this tutorial, we investigate two important properties of functions which are continuous on a closed interval **[a, b]**: the Intermediate Value Theorem and the Extreme Value Theorem. As an application of the Intermediate Value Theorem, we discuss the existence of roots of continuous functions and the bisection method for finding roots.

After working through these materials, the student should be able to apply the Intermediate Value Theorem, to determine whether certain functions have roots, and use the bisection method to find roots to any degree of accuracy.

Modules:

> **Intermediate Value Theorem.** Let **f** be a function which is continuous on the closed interval **[a, b]**. Suppose that **d** is a real number between **f(a)** and **f(b)**; then there exists **c** in **[a, b]** such that **f(c) = d**.

- **Discussion** [Using Flash]
- One of the useful consequences of the Intermediate Value Theorem is the following.

> **Corollary.** Let **f** be a function which is continuous on the closed interval **[a, b]**. Suppose that the product **f(a) f(b) < 0**; then there exists **c** in **(a, b)** such that **f(c) = 0**. In other words, **f** has at least one root in the interval **(a, b)**

- **Example.**
  - Show that $f(x) = x^3 + 4x + 4$ has a root.
    [Solution]
- Using this Corollary, we can develop an algorithm for finding roots of functions to any degree of accuracy. This algorithm is called the **Bisection Method**.

  Discussion [Using Flash]
- **Interactive Javascript module** illustrating the use of the Bisection Method.

> **Definition.** Suppose that **a** is in the domain of the function **f** such that, for all **x** in the domain of **f**,
>
> $$f(x) \leq f(a)$$
>
> then **f** is said to have **a maximum value** at **x = a**.
>
> Suppose that **a** is in the domain of the function **f** such that, for all **x** in the domain of **f**,

$$f(x) \geq f(a)$$

then **f** is said to have **a minimum value** at **x = a**.

- **Discussion** [Using Flash]
- Another very important property of continuous functions defined on closed intervals is the following theorem.

  Extreme Value Theorem. Suppose that **f** is a function which is continuous on the closed interval **[a, b]**. Then there exist real numbers **c** and **d** in **[a, b]** such that

  - **f** has a maximum value at **x = c** and
  - **f** has a minimum value at **x = d**.

- While the Extreme Value Theorem may seem intuitively obvious, it is a difficult theorem to prove. The proof is usually covered in an advanced calculus or analysis class.

  Techniques for finding **c** and **d** will be given after we develop more tools. Except for a few examples, we will rely now on graphical evidence for finding **c** and **d**.

- **Examples:**
  - $f(x) = 2 - 3x$ where $-5 \leq x \leq 8$
    [Discussion]
  - $g(x) = \sin(x)$ where $0 \leq x \leq 2\pi$
    [Discussion]
- A quiz on using concepts involving continuity of functions and its consequences.

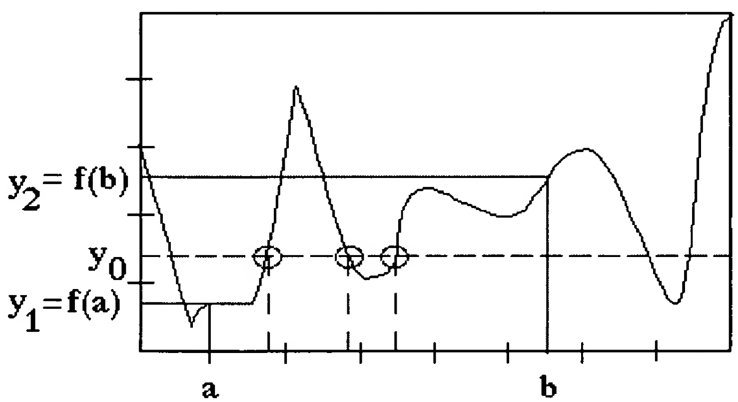
The Intermediate Value Theorem

The Intermediate Value Theorem (often abbreviated as IVT) says that if a continuous function takes on two values y_1 and y_2 at points a and b, it also takes on every value between y_1 and y_2 at some point between a and b. A typical argument using the IVT is:

Know: $e^0 = 1$ and $e^1 = e$,
$1 < 2 < e$,
e^x is continuous for all real x.

Conclude: $e^x = 2$ for some x with $0 \leq x \leq 1$.

Graphically The IVT says that if $y_1 = f(a)$ and $y_2 = f(b)$ for a function f(x), and if we draw the horizontal line $y = y_0$ for any y_0 between y_1 and y_2, then the horizontal line intersects the graph of y = f(x) at (at least one) point whose x-coordinate is between a and b.



The formal statement of the IVT is:

The Intermediate Value Theorem

Let f(x) be a function which is continuous on the closed interval [a,b] and let y_0 be a real number lying between f(a) and f(b), i.e. with $f(a) \leq y_0 \leq f(b)$ or $f(b) \leq y_0 \leq f(a)$.

Then there is at least one c with $a \leq c \leq b$ such that $y_0 = f(c)$.

Note: "continuous on the closed interval [a.b]" means that f(x) is continuous at every point x with a < x < b and that f(x) is right-continuous at x = a and left-continuous at x=b.

Example: Show that there is some u with 0 < u< 2 such that $u^2 + \cos(\pi u) = 4$.

To do this we apply the IVT to the function $h(u) = u^2 + \cos(\pi u)$. h(u) is the sum of two functions which we know are continuous (these are Field Guide functions), so h(u) is continuous. We can compute that

$$h(0) = 0^2 + \cos(0*\pi) = 1.$$
$$h(2) = 2^2 + \cos(2\pi) = 5.$$

Since $h(0) = 1 < 4 < 5 = h(2)$, and since h is continuous, the IVT implies that $h(u) = 4$ for some u in the closed interval [0,2]. This u cannot be one of the endpoints, so $0 < u < 2$.

In order to understand the IVT, we should take a closer look at

- What the IVT says and does not say.
- Why continuity is a necessary hypothesis.

This is done on the next page. Thereafter we give an application of the IVT to root finding.

/Stage4/Lesson/IVT.html

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